

# Small Entangled Quantum Worlds with a Simple Structure

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We introduce the notion of a small quantum world (SQW), which in our opinion, is very helpful in the situations when an experimenter's tools for preparing quantum states and (or) for measuring the observables of a quantum system under study are restricted due to some kind of reasons. In this case it is advisable to use as original an appropriate subspace of complete Hilbert space of states and respectively to utilize observables acting only in this subspace. If this subspace possesses some additional symmetries, the structure of a pure states set, as a rule, is simpler by far than in general case. Moreover in such SQWs some specific irreducible entangled states may appear. The similar states could be very helpful in various tasks connected with quantum informational applications. In the present paper main ideas outlined above are developed in detail in the simple and instructive case of a two-qubit system in which the accessible space of states possesses the additional symmetry structure of permutation group of three elements.

PACS numbers:

It is well known that there are two fundamental concepts in quantum theory, namely states and observables, and respectively an experimenter has to deal with two main procedures that realize these concepts, namely preparation of the required state and measuring the observable of interest in the end of the experiment. Note that among all states of quantum system its pure states, which contain the maximal possible information about the system, play the most important role. One of the main postulates of quantum theory claims that there is exact mapping between pure states of a quantum system under study and vectors of appropriate Hilbert space. For example, in the case of the most simple quantum system, that is a qubit, the relevant Hilbert space, representing its states, is two-dimensional and there is a geometrically descriptive image of the states of such system by means of the Bloch sphere. As is well known, an arbitrary state of the qubit (mixed or pure) with a density matrix  $\hat{\rho}$  can be represented in the form  $\hat{\rho} = \frac{1+\vec{P}\vec{\sigma}}{2}$ , where  $\hat{\sigma}_i$  ( $i = 1, 2, 3$ ) are the Pauli matrices and  $\vec{P}$  is appropriate Bloch vector. The pure states of a qubit for which the condition  $\hat{\rho}^2 = \hat{\rho}$  is satisfied are placed on the surface of the Bloch sphere with  $\vec{P} = 1$  while all the rest (mixed) states are settled inside this sphere. Unfortunately, in more complex situations, in particular for composite quantum systems the problem of description of the set of pure states by geometrically distinct way is unsolved till now. Furthermore, when one is operating with the states of composite systems, the important problem of determination their entanglement (the quantity that just specifies the nonlocal informational resource and distinguishes quantum communicational systems from classical ones), can be solved exactly also only for two-qubit systems. In spite of enormous number of papers devoted to this problem (see for example the comprehensive re-

view of the topic [1]), this problem remains open up to now. In this connection we propose as a first step to consider more simple problem, namely, to study some special classes of quantum systems in which both the set of accessible quantum states and the set of observables that are available for measurement are restricted by some additional conditions. Evidently we will talking about specific subspaces of the total Hilbert space and appropriate algebras of observables acting in these subspaces. From physical point of view it means that, due to various reasons, capabilities of an experimenter for the quantum states preparation and for performing arbitrary measurements are restricted. Nevertheless, since in this approach the space of accessible states of the system remains linear and closed, all postulates of quantum theory continue to be valid. Henceforth we will define as a small quantum world (SQW) certain subspace of states within the complete Hilbert space with the relevant algebra of observables acting in this subspace. The main goal of the present paper is to demonstrate on particular examples that the structure of states in such SQWs could be simpler by far than the structure of states in the enveloping large quantum world. At first let us briefly remind one of the known examples of SQW, namely, so called X-states in two-qubit quantum systems [2]. In this case one assumes that density matrix of any accessible state of the quantum system under study can be represented

as  $\hat{\rho} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}$ . The relevant basis of the

algebra of observables acting on these states consists of 8 operators (generators of the algebra). Let us write them explicitly: 1)  $\hat{1}$ —the unit operator, 2) another diagonal op-

erator  $\hat{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ , and in addition six opera-

tors  $\hat{\lambda}_i$  and  $\hat{\tau}_i$  ( $i = 1, 2, 3$ ) that are defined by analogy

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with the known Pauli matrices,namely :

$$\hat{\lambda}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \hat{\lambda}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\hat{\tau}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hat{\tau}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\tau}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to point out the complete system of algebraic relations connecting the generators of this algebra, namely:

$$\begin{aligned} \hat{\lambda}_i \hat{\lambda}_j &= \frac{1 + \hat{E}}{2} \delta_{ij} + i \varepsilon_{ijk} \hat{\lambda}_k, \\ \hat{\tau}_i \hat{\tau}_j &= \frac{1 - \hat{E}}{2} \delta_{ij} + i \varepsilon_{ijk} \hat{\tau}_k, \\ \hat{\lambda}_i \hat{\tau}_j &= \hat{\tau}_j \hat{\lambda}_i = 0, \\ \hat{E} \hat{\lambda}_i &= \hat{\lambda}_i \hat{E} = \hat{\lambda}_i \text{ and } \\ \hat{E} \hat{\tau}_i &= \hat{\tau}_i \hat{E} = -\hat{\tau}_i \end{aligned} \quad (1)$$

( where  $\varepsilon_{ijk}$  is completely antisymmetric tensor).Any X-state (that is its density matrix) can be represented as  $\hat{\rho} = \frac{1+e\hat{E}+P_i\hat{\lambda}_i+S_i\hat{\tau}_i}{4}$ .(where  $e, P_i, S_i$  are appropriate numerical coefficients). It is easy to see that 4 eigenvalues of such matrix can be calculated explicitly and are equal to:  $\rho_{1,2} = \frac{1+e\pm|P|}{4}$  and  $\rho_{3,4} = \frac{1-e\pm|S|}{4}$ . Evidently, that one can specify two disjoint classes of pure X-states in such SQW: 1) with  $e = 1$ ,  $|P| = 2$ ,  $|S| = 0$  and 2) with  $e = -1$ ,  $|S| = 2$ ,  $|P| = 0$ . Thus the structure of pure states in this SQW turns out to be simpler by far than for two-qubit states in general case. We will continue to study the properties of X- states elsewhere, while the present paper will be devoted to another interesting case, namely the SQWs that can be constructed on the basis of the permutation group of four elements-  $S_4$ . It is well-known that due to the principle of indistinguishability the permutation group plays the fundamental and diverse role in quantum theory, but in this paper it will be used only for the description of quantum states and their properties. With reference to group  $S_4$  it should be noted that among its 30 subgroups there are 4 subgroups that are isomorphic to group  $S_3$  (that is permutation group of 3 elements).Exactly this subgroup can serve as demonstrative and instructive example of the small (and entangled

as we see further) quantum world. Let us consider now the concrete realization of this subgroup which leaves as invariant the fourth element (state) of the group and construct the model of SQW based on this realization. In the standard basis of two-qubit states the relevant algebra of observables acting in this SQW consists of six generators, from which three (aside from unit operator) are Hermitian and the rest two are unitary. Let us write down them in explicit form. The three Hermitian operators are:

$$\begin{aligned} \hat{H}_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \hat{H}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \hat{H}_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (2)$$

and two unitary ones are

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \hat{B} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3)$$

Note that unitary operators  $\hat{A}$  and  $\hat{B}$  are conjugate and reciprocal to each other, that is  $\hat{A} = \hat{B}^+$  and  $\hat{A}\hat{B} = \hat{B}\hat{A} = \hat{1}$ . Let us point out the complete system of relations for hermitian generators of the algebra:  $\hat{H}_i^2 = \hat{1}$  ( $i = 1, 2, 3$ ),  $\hat{H}_1\hat{H}_2 = \hat{H}_2\hat{H}_3 = \hat{H}_3\hat{H}_1 = \hat{A}$  and  $\hat{H}_1\hat{H}_3 = \hat{H}_2\hat{H}_1 = \hat{H}_3\hat{H}_2 = \hat{B}$ . In addition let us give also the algebraic relations connecting operators  $\hat{H}_i$  with unitary operators  $\hat{A}$  which have the next form:  $\hat{H}_1\hat{A} = \hat{H}_2$ ,  $\hat{H}_2\hat{A} = \hat{H}_3$ ,  $\hat{H}_3\hat{A} = \hat{H}_1$  and  $\hat{A}\hat{H}_1 = \hat{H}_3$ ,  $\hat{A}\hat{H}_2 = \hat{H}_1$ ,  $\hat{A}\hat{H}_3 = \hat{H}_2$ . The similar equations including the matrix  $\hat{B}$  can be obtained from these relations by conjugation. There are also two relations, connecting operators  $\hat{A}$  and  $\hat{B}$  :  $\hat{A}^2 = \hat{B}$ , and  $\hat{B}^2 = \hat{A}$ . The density matrix of any state belonging to this SQW may be represented as  $\hat{\rho} = \frac{k}{2}\hat{1} + l\hat{H}_1 + m\hat{H}_2 + n\hat{H}_3 + p(\hat{A} + \hat{B})$ , where  $k, l, m, n, p$  are real numbers satisfying the normalization condition:  $k + l + m + n + p = \frac{1}{2}$ . However, it is easy to see that generators of the algebra  $\hat{H}_i$ ,  $\hat{A}$  and  $\hat{B}$  are connected by the additional relation :  $\hat{A} + \hat{B} = \hat{C} - 1$ , where  $\hat{C} \equiv \hat{H}_1 + \hat{H}_2 + \hat{H}_3$ . Thus the general representation for the density matrix in given SQW may be written finally as :

$$\hat{\rho} = \frac{a}{2} + b\hat{H}_1 + c\hat{H}_2 + d\hat{H}_3 \quad (4)$$

with normalization condition:  $a + b + c + d = \frac{1}{2}$ .

It is worth noting that operator  $\hat{C}$  commutes with all generators of the algebra and hence is the Casimir operator of the group  $S_3$ . Let us turn now to the question of clarifying the structure of pure states in this SQW. Starting from the representation (4) and using the defining condition for pure states:  $\hat{\rho}^2 = \hat{\rho}$ , one, after elementary

calculations, can obtain the following restrictions on the coefficients of the decomposition (4), namely:

$$\begin{aligned}\frac{a^2}{2} &= \frac{a^2}{4} + b^2 + c^2 + d^2 - (bc + bd + cd); \\ b &= ab + (bc + bd + cd); \\ c &= ac + (bc + bd + cd); \\ d &= ad + (bc + bd + cd).\end{aligned}\quad (5)$$

It is clear that the only possibility to satisfy all relations (5) is to put the coefficient  $a$  equal to unit and impose on the coefficients  $b, c, d$  the next restriction:  $b^2 + c^2 + d^2 = \frac{1}{4}$  (together with normalization condition  $b + c + d = -\frac{1}{2}$ ). Thus, in the parameter space of coefficients  $b, c, d$  the set of pure states is the intersection of the sphere centered in the origin, whose radius is equal to  $\frac{1}{2}$ , and the plane satisfying the equation  $b + c + d = -\frac{1}{2}$ . Evidently it is a circle. Further, it is well-known that in Hilbert space pure states form the boundary of a convex set of all quantum states [3]. Hence the result obtained means that all mixed states of the system in this SQW must be settled within the circle specified above. Thus the structure of quantum states in the SQW under study is quite simple and obvious. In addition one can point out the parametrization of pure states in this SQW by writing the coefficients of (4) with  $a = 1$  in the next convenient form:

$$\begin{aligned}b &= -\frac{t(1+t)}{2(1+t+t^2)}; c = -\frac{(1+t)}{2(1+t+t^2)}; \\ d &= \frac{t}{2(1+t+t^2)}\end{aligned}\quad (6)$$

(with the only real parameter  $t$ ). It is easy to verify directly that two relations  $b + c + d = -\frac{1}{2}$  and  $b^2 + c^2 + d^2 = \frac{1}{4}$  characterizing pure states in this SQW are fulfilled automatically. Using the parametrization (6) one can write down the normalized vector  $|\Psi\rangle$  corresponding to the density matrix of a pure state, that is, if  $\hat{\rho} = |\Psi\rangle\langle\Psi|$ , then the appropriate vector  $|\Psi\rangle$  can be represented as:

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2(1+t+t^2)}} \begin{pmatrix} 1+t \\ t \\ 1 \\ 0 \end{pmatrix}.$$

Since all information contained in quantum state of the system can be extracted only by making appropriate measurements, it is useful to give a simple experimental criterion of the state purity. To this end we assume that the unknown quantum state has the above-mentioned form:

$$\begin{aligned}\hat{\rho} &= \frac{1}{2} + b\hat{H}_1 + c\hat{H}_2 + d\hat{H}_3 = \\ &= \begin{pmatrix} \frac{1}{2} + d & b & c & 0 \\ b & \frac{1}{2} + c & d & 0 \\ c & d & \frac{1}{2} + b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}\quad (7)$$

First of all let us find the eigenvalues of the density matrix (7). One can verify easily that  $\hat{\rho}$  has two zero eigen-

values: the first with eigenvector  $|0\rangle_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$  and the

second with eigenvector  $|0\rangle_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ . Besides them

there are two nonzero eigenvalues  $\mu_1$  and  $\mu_2$  that satisfy to the quadratic equation:  $\mu^2 - \mu + 3(bc + bd + cd) = 0$  with the solutions:

$$\mu_{1,2} = \frac{1 \pm \sqrt{1 - 12(bc + bd + cd)}}{2}\quad (8)$$

The expression (8) implies that coefficients  $b, c, d$  aside from normalization condition must satisfy to the inequality  $0 \leq (bc + bd + cd) \leq \frac{1}{12}$ .

Let us define now as usual the mean value of an arbitrary observable  $\hat{A}$  as  $\langle\hat{A}\rangle = \text{Tr}(\hat{\rho}\hat{A})$ . Using the representation (7) one can write down the mean values of the following three selected observables:

- 1)  $\langle H_1 \rangle - 1 \equiv A_1 = c + d + 4b$ ,
- 2)  $\langle H_2 \rangle - 1 \equiv A_2 = b + d + 4c$ , and
- 3)  $\langle H_3 \rangle - 1 \equiv A_3 = b + c + 4d$ .

Taking into account the condition  $b + c + d = -\frac{1}{2}$  one can find that  $\langle A_1 + A_2 + A_3 \rangle = -3$ , that is for all states of SQW (with  $a = 1$ ) the mean value of the Casimir operator  $\langle C \rangle \equiv \langle H_1 + H_2 + H_3 \rangle = 0$ . On the other hand if one considers another useful quantity, namely  $R \equiv A_1^2 + A_2^2 + A_3^2 = (c + d + 4b)^2 + (b + d + 4c)^2 + (b + c + 4d)^2$ , then she(he) obtains that  $R = (-\frac{1}{2} + 3b)^2 + (-\frac{1}{2} + 3c)^2 + (-\frac{1}{2} + 3d)^2 = 9(\frac{1}{4} + b^2 + c^2 + d^2) = \frac{9}{2}$ . Thus, for all pure states in SQW the following criterion of purity should be true:

$$(\langle H_1 \rangle - 1)^2 + (\langle H_2 \rangle - 1)^2 + (\langle H_3 \rangle - 1)^2 = \frac{9}{2}\quad (9)$$

Let us go to the next question which we are interested in: how the explicit expression for the entanglement of the states (both pure and mixed) in this SQW looks? For the sake of simplicity as before we are limited ourselves to studying the states for which the coefficient  $a$  in (4) is equal to unit and the appropriate density matrix can be represented as

$$\hat{\rho} = \frac{1}{2} + b\hat{H}_1 + c\hat{H}_2 + d\hat{H}_2\quad (10)$$

with a normalization condition:  $b + c + d = -\frac{1}{2}$ . As was explained above, among these states there are certain pure states (for which  $b^2 + c^2 + d^2 = \frac{1}{4}$ ) while all the rest are mixed. Let us determine the entanglement of the state (10). To this end in the case of two-qubit arbitrary mixed state the well-known recipe was proposed by Wootters in [4]. This recipe reads as follows. First of all one must construct the auxiliary matrix

$\hat{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ , where  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$  is the Pauli matrix and  $\hat{\rho}^*$  is the matrix conjugated to given matrix  $\hat{\rho}$  (10). At the next step one needs to introduce another auxiliary matrix  $\hat{\Omega} = \hat{\rho} \hat{\rho}^*$  (that is non-Hermitian and positive) and after that to find its four eigenvalues  $\Omega_i$  ( $i = 1, 2, 3, 4$ ). If one arranges them in decreasing order  $\Omega_1 \geq \Omega_2 \geq \Omega_3 \geq \Omega_4 \geq 0$ , then according to the paper [4] the entanglement of formation  $E(\hat{\rho})$  for the state  $\hat{\rho}$  may be calculated as follows:

$$E(\hat{\rho}) = -\frac{(1 + \sqrt{1 - C^2})}{2} \log_2 \frac{1 + \sqrt{1 - C^2}}{2} - \frac{(1 - \sqrt{1 - C^2})}{2} \log_2 \frac{(1 - \sqrt{1 - C^2})}{2}, \quad (11)$$

where the concurrence

$$C = C(\hat{\rho}) = \max \left\{ 0, \sqrt{\Omega_1} - \sqrt{\Omega_2} - \sqrt{\Omega_3} - \sqrt{\Omega_4} \right\}.$$

Note that since  $E(\hat{\rho})$  is a monotonic and increasing function of concurrence  $C(\hat{\rho})$ , we can restrict ourselves to determination of the value of  $C(\hat{\rho})$  that evidently ranges from zero to unit and defines the degree of entanglement as well as  $E(\hat{\rho})$ . Omitting elementary calculations we give the final expressions for the auxiliary matrix  $\hat{\Omega}$  and its four eigenvalues, namely:

$$\hat{\Omega} = \begin{pmatrix} 0 & b(\frac{1}{2} + b) + cd & bd + c(\frac{1}{2} + c) & -2bc \\ 0 & (\frac{1}{2} + b)(\frac{1}{2} + c) + d^2 & 2(\frac{1}{2} + c)d & 0 \\ 0 & 2(\frac{1}{2} + b)d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (12)$$

The eigenvalues of matrix  $\hat{\Omega}$  (in decreasing order) are equal to:  $\Omega_1 = \left[ d + \sqrt{(\frac{1}{2} + b)(\frac{1}{2} + c)} \right]^2$ ,  $\Omega_2 = \left[ d - \sqrt{(\frac{1}{2} + b)(\frac{1}{2} + c)} \right]^2$ ,  $\Omega_3 = \Omega_4 = 0$ .

Using the above-mentioned recipe for determination of entanglement one can find the required result:

$$C(\hat{\rho}) = 2\sqrt{\left(\frac{1}{2} + b\right)\left(\frac{1}{2} + c\right)}. \quad (13)$$

It is worth noting that if the state (6) is pure, that is

$$\hat{\rho} = |\Psi\rangle\langle\Psi|, \text{ where } |\Psi\rangle = \frac{1}{\sqrt{1+t+t^2}} \begin{pmatrix} 1+t \\ t \\ 1 \\ 0 \end{pmatrix}, \text{ then the}$$

expression (13) takes the form:  $C\{\Psi\} = \frac{|t|}{1+t+t^2}$  which coincides with the standard definition of concurrence of a pure state  $|\Psi\rangle$ . We see that unlike of general case the entanglement of states belonging to the SQW considered may be explicitly expressed in terms of the coefficients of the given density matrix only. Now we turn to the study another interesting problem, namely, to clarify how much entanglement may be extracted from the given state (pure or mixed) belonging to the SQW by means

of various measurements carried out on a system being in the state (10). Clearly, having in hands the simple expression for the amount of entanglement contained in any quantum state of SQW (13), this problem may be easily solved. We consider here only the cases when the measured observables are the basic observables that is generators of the algebra  $H_1, H_2, H_3$

Note that in view of relation  $\hat{H}_i^2 = 1$  (for all  $i = 1, 2, 3$ ) one can write a simple equation connecting two density matrices, namely, density matrix  $\hat{\rho}_0$  before the measurement and the density matrix  $\hat{\rho}_\infty$  after the measurement of the observable  $\hat{H}_i$ . This equation reads as:

$$\hat{\rho}_\infty = \frac{\hat{\rho}_0 + \hat{H}_i \hat{\rho}_0 \hat{H}_i}{2}. \quad (14)$$

Using Eq.(14) and the algebra of operators described above one can consider separately three cases: the case I when the observable  $\hat{H}_1$  is measured and the initial state is  $\hat{\rho}_0 = \frac{1}{2} + b\hat{H}_1 + c\hat{H}_2 + d\hat{H}_3$ , the case II when the observable  $\hat{H}_2$  is measured and the case III when observable  $\hat{H}_3$  is measured. In the case I after the measurement the state of the system is:  $\hat{\rho}_\infty^I = \frac{1}{2} + b\hat{H}_1 + \frac{(c+d)}{2}(\hat{H}_2 + \hat{H}_3)$ . Similarly in the case II the final state is  $\hat{\rho}_\infty^{II} = \frac{1}{2} + \frac{(b+d)}{2}(\hat{H}_1 + \hat{H}_3) + c\hat{H}_2$  and in the case III the final state of the system after measurement is  $\hat{\rho}_\infty^{III} = \frac{1}{2} + \frac{(b+c)}{2}(\hat{H}_1 + \hat{H}_2) + d\hat{H}_3$ . Now we are interested in the maximum amount of entanglement that can be extracted from initial state by these different measurements. Let us calculate this maximum. Without loss of generality we may assume that the initial state of the system is pure and can be represented by parametrization (6). Then the gain of entanglement caused by measurement of  $\hat{H}_1$  may be written as:

$$\begin{aligned} \Delta C_I &= C\{\hat{\rho}_\infty^I\} - C\{\hat{\rho}_0\} = \\ &= 2\sqrt{\left(b + \frac{1}{2}\right)\left(\frac{c+d+1}{2}\right)} - 2\sqrt{\left(b + \frac{1}{2}\right)\left(c + \frac{1}{2}\right)} = \\ &= \frac{1}{1+t+t^2} \left[ \sqrt{\frac{1+2t+2t^2}{2}} - |t| \right]. \end{aligned} \quad (15)$$

Note that in the derivation of relation (15) we used the parametrization (6) for coefficients  $b, c, d$  of the initial pure state. It is easy to see that maximum (15) is equal to  $\frac{1}{\sqrt{2}}$  and is reached when  $t = 0$ . The required initial

state in this case is  $|\Psi_0\rangle_I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ . In the same way

one can find that in the case II the gain of entanglement can be represented as:

$$\Delta C_{II} = \frac{|t|}{1+t+t^2} \left[ \sqrt{\frac{2+2t+t^2}{2}} - 1 \right]. \quad (16)$$

The maximum of (16) is reached when  $t = \pm\infty$  and is equal to  $\frac{1}{\sqrt{2}}$  as well. The required initial state in this case

is  $|\Psi_0\rangle_{II} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ 0 \end{pmatrix}$ . However, in the case III, when

the observable  $\hat{H}_3$  is measured, the result is somewhat distinct, namely, the gain of entanglement caused by this measurement can be written as:

$$\Delta C_{III} = C\{\hat{\rho}_\infty^{III}\} - C\{\hat{\rho}_0\} = \frac{1+t^2-2|t|}{2(1+t+t^2)}. \quad (17)$$

It is easy to see that the maximum of (17) is reached when  $t = 0$  and is equal to  $\frac{1}{2}$ . It is curious that, although optimal initial states for cases I and III coincide, nevertheless, the extracted amount of entanglement is larger in the first case. Now let us consider another important and interesting feature of certain states belonging to the SQWs that makes them potentially very helpful in various quantum informational applications. We have in mind the existence of irreducible entangled states both in SQW under examination and in many others SQWs as well. Really let us consider the selected mixed state *IE* (that is irreducible and entangled) with density matrix  $\hat{\rho}_{IE} = \frac{1}{2} - \frac{\hat{C}}{6} \equiv \frac{1}{2} - \frac{\hat{H}_1 + \hat{H}_2 + \hat{H}_3}{6}$ . It is easy to see that this state belongs to the SQW because two necessary conditions:  $b + c + d = -\frac{1}{2}$  and  $bc + bd + cd = \frac{1}{12} \leq \frac{1}{12}$  are satisfied. On the other hand, it is clear that this state is irreducible (that is cannot be changed in time by any

dynamical way or by means of measurements) because the Casimir operator  $\hat{C} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3$  commutes with all generators of the algebra.

In addition note that this selected state is entangled with concurrence  $C_{IE} = \frac{2}{3}$ . Thus we come to the conclusion that similar states (in the case when realization of the SQW would be possible) may be used as long-lived keepers of entanglement stored in the system. The mentioned feature of irreducible entangled states makes them indispensable for various quantum informational applications. It should also be noted that although the IE state is dynamically stable it can be achieved easily from other states by means of appropriate measurements. For example if one takes the initial state of the system in the form:  $\hat{\rho}_0 = \frac{1}{2} - \frac{\hat{H}_1}{6} + c\hat{H}_2 + d\hat{H}_3$  and then performs the measurement of the observable  $\hat{H}_1$  in this state then (if the conditions: 1)  $c + d = -\frac{1}{3}$  and 2)  $cd \leq \frac{5}{36}$  hold true) she(he) gets exactly the required IE state after the measurement.

Let us sum up the main results obtained in the present paper. We introduce the notion of small quantum world (SQW) which allows one to study special classes of composite quantum systems whose pure states and nonlocal properties turn out to be simpler by far comparing with large quantum worlds containing them as a small part. We are demonstrating that some features of selected states belonging to these SQWs would be very helpful for performing various quantum information tasks.

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